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Existence and attractivity of global solutions for a class of fractional quadratic integral equations in Banach space

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Abstract

In this paper, we present some results concerning the existence and attractivity of global solutions for a class of nonlinear fractional integral equations and fractional differential equations in a Banach space X, respectively. These results are new even in the case of $X = \mathbf{R}$. Some examples are given to show the applications of the abstract results.

MSC: 34K05; 34A12

Keywords: fractional order; integral equations; fractional differential equations; global solutions; attractivity

1 Introduction

The theory of integral equations is frequently applicable in other branches of mathematics and mathematical physics, like as radiative transfer, kinetic theory of gases, neutron transport, *etc.*, and engineering, economics, biology as well as in describing problems connected with the real world (see [1–3] and the references therein). On the other hand, the application of fractional calculus is very broad, including the characterization of mechanics and electricity, earthquake analysis, the memory of many kinds of material, electronic circuits, electrolysis chemical, *etc.* [4–6].

Recently, some basic theory for initial value problems for fractional differential equations and inclusions was discussed by many researchers (see [7–10] and the references therein). Moreover, there has been a significant development in solving integral equations involving fractional derivatives in $X = \mathbf{R}$ (see [11–16] and the references therein). However, to the best of our knowledge, there are few works on the attractivity of solutions for fractional integral equations in a Banach space.

In this paper, let X be a Banach space, we discuss the existence and attractivity of global solutions for fractional integral equation on X of the form

$$x(t) = f\bigl(t, x\bigl(\alpha(t)\bigr)\bigr) + \frac{g(t, x(\beta(t)))}{\Gamma(q)} \int_0^{\eta(t)} \frac{K(t,s)h(s, x(\gamma(s)))}{(t-s)^{1-q}}\, ds, \quad t \in [0, +\infty), \tag{1.1}$$

where $0 < q < 1$. $f, g, h : \mathbf{R}_+ \times X \to X$ are to be specified later. $\alpha, \beta, \gamma, \eta : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous. $\{K(t,s) : t \geq s, t, s \in \mathbf{R}_+\}$ is a set of bounded linear operators on X.

In previous work [11–16], there are different techniques dealing with integral equations. For example, the authors apply the measure of noncompactness and fixed point theorem of Darbo type to deal with quadratic Erdélyi-Kober type integral equations of fractional order with three parameters [11]; the authors deal with integral equations in an order interval of a cone [12]; in [14], the technique used is the measure of noncompactness on space $BC(\mathbf{R}^+)$ associated with the Schauder fixed point theorem, *etc.* In [14–16] and the references therein, the continuity (even Lipschitz continuity) of f (or g, h) of two variables is always required. In this paper, we study a more complicated model than previous ones [14–16], apply different techniques and obtain results under weaker assumptions. We define a Banach space $C^0_\delta(X)$ and investigate the existence and attractivity of global solutions for equation (1.1) on $C^0_\delta(X)$ by using the fixed point theorem. The above mentioned techniques are new even in the case of $X = \mathbf{R}$.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and preliminary facts. In Section 3, we present the notion of the so-called space $C^0_\delta(X)$ and study equation (1.1) on this space with the fixed point theorem, as the additional production, we obtain the corresponding results for the global mild solutions of the fractional differential equations as follows:

$$\begin{cases} {}^cD^q_t(x(t) - m(t,x(t))) = A(x(t) - m(t,x(t))) + h(t,x(t)), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1.2}$$

and the global solutions of the following fractional differential equations:

$$\begin{cases} {}^cD^q_t x(t) = h(t,x(t)), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ in X. Finally, three examples are given to illustrate our main results.

2 Preliminaries

We introduce some terminology. Throughout this paper, X denotes a Banach space with norm $\|\cdot\|$ and $\mathcal{L}(X)$ the Banach space of all linear and bounded operators on X. We write $B_r(x,Z)$ to denote the closed ball with center at x and radius $r > 0$ in a Banach space Z, and $C(\mathbf{R}_+,X)$ the space of all X-valued continuous functions on $\mathbf{R}_+$ with the supremum norm $\|x\|_\infty = \sup\{\|x(t)\| : t \geq 0\}$ for any $x \in C(\mathbf{R}_+,X)$.

We recall the following basic definitions and properties of the fractional calculus theory. For more details see [5].

Definition 2.1 ([5]) The fractional integral of order q with the lower limit zero for a function $f \in L^1[0,\infty)$ is defined as

$$J^q_t f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds, \quad t > 0, 0 < q < 1,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([5]) The Riemann-Liouville derivative of order q with the lower limit zero for a function $f \in C^1[0,\infty)$ can be written as

$$^{\mathrm{L}}D_t^q f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q} f(s)\,ds, \quad t > 0, 0 < q < 1.$$

Definition 2.3 ([5]) The Caputo derivative of order q for a function $f \in C^1[0,\infty)$ can be written as

$$^{\mathrm{c}}D_t^q f(t) = {}^{\mathrm{L}}D_t^q\bigl(f(t) - f(0)\bigr), \quad t > 0, 0 < q < 1,$$

where $^{\mathrm{c}}D_t^q := \frac{d^q}{dt^q}$. Moreover, $^{\mathrm{c}}D_t^q c = 0$, where c is a constant.

Remark 2.4 ([5]) If $\alpha \in (0,1)$, $f \in C[0,\infty)$, then $(J_t^{\alpha\,\mathrm{c}}D_t^\alpha f)(t) = f(t) - f(0)$ and $({}^{\mathrm{c}}D_t^\alpha J_t^\alpha f)(t) = f(t)$.

We need the following lemma concerning on a fixed point theorem (see [17], Theorem 4.3.2).

Lemma 2.5 *Let D be a convex, bounded, and closed subset of a Banach space Z and $F: D \to D$ be a condensing map. Then F has a fixed point in D.*

Assuming that Ω is a nonempty subset of the space $C(\mathbf{R}_+, X)$, we review the concept of attractivity of solutions of equation (1.1).

Definition 2.6 ([17]) We say that solutions of equation (1.1) are *locally attractive* if there exists a ball $B(x_0, C(\mathbf{R}_+, X))$ such that for arbitrary solutions $x(t)$ and $y(t)$ of equation (1.1) belonging to $B(x_0, C(\mathbf{R}_+, X)) \cap \Omega$,

$$\lim_{t\to\infty} \bigl\|x(t) - y(t)\bigr\| = 0 \tag{2.1}$$

holds.

Remark 2.7 When the limit (2.1) is uniform with respect to the set $B(x_0, C(\mathbf{R}_+, X)) \cap \Omega$, *i.e.* when for each $\varepsilon > 0$ there exists $T > 0$ such that $\|x(t) - y(t)\| < \varepsilon$ for all solutions $x, y \in B(x_0, C(\mathbf{R}_+, X)) \cap \Omega$ of equation (1.1) and for any $t \geq T$, we will say that solutions of equation (1.1) are *uniformly locally attractive* (or equivalently that solutions of (1.1) are asymptotically stable).

Lemma 2.8 ([18])

(1) *For all $\tau, \theta > -1$, we have*

$$\int_0^t s^\theta (t-s)^\tau\,ds = t^{\theta+\tau+1} B(\tau+1, \theta+1).$$

(2) *For all $\lambda, \kappa, \varpi > 0$, for $t \geq 0$, we have*

$$\int_0^t s^{\lambda-1}(t-s)^{\kappa-1} e^{-\varpi(t-s)}\,ds \leq \max\bigl\{1, 2^{1-\lambda}\bigr\}\Gamma(\kappa)\left(1 + \frac{\kappa}{\lambda}\right)\varpi^{-\kappa} t^{\lambda-1},$$

where $B(\cdot,\cdot)$ is the beta function.

3 Main results

In this section, we study the existence and attractivity of global solutions for equation (1.1), equation (1.2), and equation (1.3), respectively.

3.1 The case of fractional integral equations

Motivated by the work in [19], for any $\delta > 0$, we define the space

$$C_\delta^0(X) = \left\{ x \in C(\mathbf{R}_+, X) : \lim_{t\to\infty} \frac{\|x(t)\|}{e^{\delta t}} = 0 \right\}$$

endowed with the norm $\|x\|_\delta = \sup_{t\geq 0} e^{-\delta t}\|x(t)\|$.

We recall the following results of compactness of this spaces.

Lemma 3.1 ([19]) *A set $B \subset C_\delta^0(X)$ is relatively compact in $C_\delta^0(X)$ if and only if*:

(a) *B is equicontinuous*;
(b) $\lim_{t\to\infty} e^{-\delta t}\|x(t)\| = 0$, *uniformly for $x \in B$*;
(c) *the set $B(t) = \{x(t) : x \in B\}$ is relatively compact in X, for every $t \geq 0$*.

Let $r > 0$, and in the following let $V(r,g)$ denote the set $V(r,g) = \{t \to g(t, x(\beta(t))) : x \in B_r(0, C(\mathbf{R}_+, X))\}$.

We study the fractional integral equation (1.1) with the following conditions:

(H1) $\alpha, \beta, \gamma, \eta : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, η is nondecreasing on $\mathbf{R}_+$, $\alpha(t) \to \infty$, $\eta(t) \to \infty$, as $t \to \infty$, and $\alpha(t), \eta(t), \gamma(t) \leq t$.

(H2) The function $f(t, x(\alpha(t)))$ is continuous with respect to t on $[0, +\infty)$ and there exists a continuous function $L_f(t)$ such that

$$\bigl\|f(t,\psi_1) - f(t,\psi_2)\bigr\| \leq L_f(t)\|\psi_1 - \psi_2\|, \quad \text{for } \psi_1, \psi_2 \in C(\mathbf{R}_+, X),$$

where $L_f^* = \sup_{t\geq 0} L_f(t) < 1$, $\lim_{t\to\infty} L_f(t) = 0$, and $\sup_{t\geq 0} \|f(t,0)\| < \infty$.

(H3) The function $g(t, x(\beta(t)))$ is continuous with respect to t on $[0, +\infty)$ and for every $a > 0$, $g(t,\cdot) : X \to X$ is continuous for $t \in [0,a]$. The set $V(r,g)$ is an equicontinuous subset of $C(\mathbf{R}_+, X)$ and there exists a function $\zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\|g(t,x)\| \leq \zeta(t)$ and $\sup_{t\in[0,a]} \zeta(t) = \tilde{\zeta}_a < \infty$, for every $a > 0$.

(H4) The operator $K(\cdot, s)$ is continuous in the uniform operator topology for all $s \in \mathbf{R}_+$ and $\overline{k} = \|K(t,s)\|_{\mathcal{L}(X)} < \infty$.

(H5) For any $a > 0$, the function $h : [0,a] \times X \to X$ satisfies the following conditions:

  (a) the function $h(t,\cdot) : X \to X$ is continuous a.e. $t \in [0,a]$;
  (b) the function $h(\cdot, x) : [0,a] \to X$ is strongly measurable for every $x \in X$;
  (c) there exists $\nu(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that $\|h(t,x)\| \leq \nu(t)\|x\|$, the function $s \to \frac{\nu(s)}{(t-s)^{1-q}}$ belongs to $L^1([0,t], \mathbf{R}_+)$ and

$$\lim_{t\to\infty} \zeta(t) \int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\, ds = 0;$$

  (d) for every $t > 0$ and $r > 0$, the set $\{K(t,s)h(s, e^{\delta s}z) : s \in [0,t), z \in B_r(0, C_\delta^0(X))\}$ is relatively compact in X.

In the following, we choose a constant $\delta > 0$ such that

$$L_f^* + \sup_{t\geq 0} \frac{\zeta(t)\overline{k}}{\Gamma(q)} \int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds < 1. \tag{3.1}$$

Theorem 3.2 *Let the assumptions* (H1)-(H5) *be satisfied, then there exists a solution for equation* (1.1) *on the space* $C_\delta^0(X)$.

Proof For $x \in C_\delta^0(X)$, we consider the operator $\mathcal{M}$ of the form

$$(\mathcal{M}x)(t) = (\mathcal{F}x)(t) + (\mathcal{G}x)(t),$$

where

$$(\mathcal{F}x)(t) = f\bigl(t, x\bigl(\alpha(t)\bigr)\bigr), \qquad (\mathcal{G}x)(t) = \frac{g(t, x(\beta(t)))}{\Gamma(q)} \int_0^{\eta(t)} \frac{K(t,s)h(s, x(\gamma(s)))}{(t-s)^{1-q}}\,ds.$$

From our assumptions, it is easy to see that $\mathcal{G}x \in C(\mathbf{R}_+, X)$ and

$$\begin{aligned} e^{-\delta t}\bigl\|(\mathcal{G}x)(t)\bigr\| &\leq \frac{\overline{k}\zeta(t)e^{-\delta t}}{\Gamma(q)} \int_0^{\eta(t)} \frac{\nu(s)e^{\delta\gamma(s)}e^{-\delta\gamma(s)}\|x(\gamma(s))\|}{(t-s)^{1-q}}\,ds \\ &\leq \frac{\overline{k}\zeta(t)}{\Gamma(q)} \int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \cdot \|x\|_\delta \to 0, \quad t \to \infty, \end{aligned}$$

and we conclude that $\mathcal{G}x \in C_\delta^0(X)$ and $\mathcal{G}$ is a function from $C_\delta^0(X)$ to $C_\delta^0(X)$.

Applying condition (H2), we get

$$\frac{\|(\mathcal{F}x)(t)\|}{e^{\delta t}} \leq \frac{1}{e^{\delta t}}\bigl[\bigl\|f\bigl(t, x\bigl(\alpha(t)\bigr)\bigr) - f(t,0)\bigr\| + \bigl\|f(t,0)\bigr\|\bigr] \leq L_f(t)\|x\|_\delta + \frac{\|f(t,0)\|}{e^{\delta t}}.$$

Hence, $\mathcal{F}$ is $C_\delta^0(X)$-valued. Moreover, $\|\mathcal{F}x - \mathcal{F}y\|_\delta \leq L_f^*\|x-y\|_\delta$, which implies that $\mathcal{F}$ is a contraction on $C_\delta^0(X)$.

Now, we show that $\mathcal{G}$ is continuous. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in $C_\delta^0(X)$ such that $x_n \to x$ in $C_\delta^0(X)$ as $n \to \infty$, that is, for arbitrary $\varepsilon > 0$ such that $\|x_n - x\|_\delta < \varepsilon$ for sufficient large n. We can see

$$\begin{aligned} &e^{-\delta t}\bigl\|(\mathcal{G}x_n)(t) - (\mathcal{G}x)(t)\bigr\| \\ &\quad\leq \frac{\overline{k}\|g(t, x_n(\beta(t))) - g(t, x(\beta(t)))\| \cdot \|x_n\|_\delta}{\Gamma(q)} \int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \\ &\qquad + \frac{\overline{k}\zeta(t)}{\Gamma(q)e^{\delta t}} \int_0^{\eta(t)} \frac{\|h(s, x_n(\gamma(s))) - h(s, x(\gamma(s)))\|}{(t-s)^{1-q}}\,ds \\ &\quad= I_1(t) + I_2(t). \end{aligned}$$

From condition (H5)(c) there exists $T > 0$, for $t \geq \eta(t) \geq T$ such that

$$\frac{\overline{k}\zeta(t)}{\Gamma(q)} \int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds < \frac{\varepsilon}{8(\varepsilon + 2\|x\|_\delta)}. \tag{3.2}$$

Moreover, noting that $\|x_n\|_\delta \le \varepsilon + \|x\|_\delta$ and from (H3), there exists $N_\varepsilon^1 \in \mathbf{N}$ such that

$$\begin{aligned}
&\bigl\|g\bigl(t,x_n\bigl(\beta(t)\bigr)\bigr)-g\bigl(t,x\bigl(\beta(t)\bigr)\bigr)\bigr\| \\
&\quad \le \frac{\varepsilon\Gamma(q)}{4\overline{k}(\varepsilon+\|x\|_\delta)\int_0^T \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds}, \quad t\in[0,T], n>N_\varepsilon^1,
\end{aligned} \tag{3.3}$$

then

$$\bigl\|I_1(t)\bigr\| < \frac{\varepsilon}{4}, \quad \text{for } t\in[0,T], n>N_\varepsilon^1, \tag{3.4}$$

on the other hand, for $t\ge\eta(t)\ge T$, $N_\varepsilon^1\in\mathbf{N}$, noting that (3.2) and (3.3), we have

$$\begin{aligned}
\bigl\|I_1(t)\bigr\| &\le \frac{\overline{k}\|g(t,x_n(\beta(t)))-g(t,x(\beta(t)))\|\cdot\|x_n\|_\delta}{\Gamma(q)}\int_0^T \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \\
&\quad + \frac{2\overline{k}\zeta(t)\|x_n\|_\delta}{\Gamma(q)}\int_T^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \\
&< \frac{\varepsilon}{2},
\end{aligned}$$

then

$$\bigl\|I_1(t)\bigr\| < \frac{\varepsilon}{2}, \quad \text{for } t\ge\eta(t)\ge T, n>N_\varepsilon^1. \tag{3.5}$$

For $I_2(t)$, from the Lebesgue dominated convergence theorem and (H5)(a), there exists $N_\varepsilon^2\in\mathbf{N}$ such that

$$\frac{\overline{k}}{\Gamma(q)}\int_0^T \frac{\|h(s,x_n(\gamma(s)))-h(s,x(\gamma(s)))\|}{(t-s)^{1-q}}\,ds < \frac{\varepsilon}{4\tilde{\zeta}_T}, \quad n>N_\varepsilon^2, \tag{3.6}$$

then we have

$$\bigl\|I_2(t)\bigr\| < \frac{\varepsilon}{4}, \quad \text{for } t\in[0,T], n>N_\varepsilon^2, \tag{3.7}$$

on the other hand, for $t\ge\eta(t)\ge T$ and $n>N_\varepsilon^2$, noting that

$$\begin{aligned}
&\bigl\|h\bigl(t,x_n\bigl(\gamma(t)\bigr)\bigr)-h\bigl(t,x\bigl(\gamma(t)\bigr)\bigr)\bigr\| \\
&\quad \le \nu(t)\bigl(\bigl\|x_n\bigl(\gamma(t)\bigr)-x\bigl(\gamma(t)\bigr)\bigr\|+2\bigl\|x\bigl(\gamma(t)\bigr)\bigr\|\bigr) \\
&\quad \le \nu(t)e^{\delta t}\bigl(\varepsilon+2\|x\|_\delta\bigr),
\end{aligned}$$

and (3.2), (3.6), we obtain

$$\begin{aligned}
\bigl\|I_2(t)\bigr\| &\le \frac{\tilde{\zeta}_T\overline{k}}{\Gamma(q)e^{\delta t}}\int_0^T \frac{\|h(s,x_n(\gamma(s)))-h(s,x(\gamma(s)))\|}{(t-s)^{1-q}}\,ds \\
&\quad + \frac{\zeta(t)\overline{k}}{\Gamma(q)e^{\delta t}}\int_T^{\eta(t)} \frac{\|h(s,x_n(\gamma(s)))-h(s,x(\gamma(s)))\|}{(t-s)^{1-q}}\,ds
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon}{4} + \frac{\zeta(t)\overline{k}}{\Gamma(q)} \int_T^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \cdot \bigl(\varepsilon + 2\|x\|_\delta\bigr) \\
&< \frac{\varepsilon}{2}, \quad t \geq \eta(t) \geq T, n > N_\varepsilon^2.
\end{aligned}
\tag{3.8}
$$

Now, from (3.4) and (3.7), we have

$$
\sup\bigl\{e^{-\delta t}\bigl\|(\mathcal{G}x_n)(t) - (\mathcal{G}x)(t)\bigr\| : t \in [0,T], n > \max\bigl\{N_\varepsilon^1, N_\varepsilon^2\bigr\}\bigr\} \leq \varepsilon,
$$

and from (3.5) and (3.8), we obtain

$$
\sup\bigl\{e^{-\delta t}\bigl\|(\mathcal{G}x_n)(t) - (\mathcal{G}x)(t)\bigr\| : t \geq \eta(t) \geq T, n > \max\bigl\{N_\varepsilon^1, N_\varepsilon^2\bigr\}\bigr\} \leq \varepsilon.
$$

Now, we can conclude that $\mathcal{G}$ is continuous.

Next, we show that $\mathcal{G}$ is completely continuous. Let $r > 0$ and $B_r = B_r(0, C_\delta^0(X))$, we first of all show that the set $\mathcal{G}(B_r)$ is equicontinuous. For any $\varepsilon > 0$, $t_1, t_2 \geq 0$, we may assume that $t_1 < t_2$ without loss of generality, for $x \in B_r$, we get

$$
\begin{aligned}
&\bigl\|(\mathcal{G}x)(t_2) - (\mathcal{G}x)(t_1)\bigr\| \\
&\quad\leq \frac{\overline{k}e^{\delta t_2} r\|g(t_2, x(\beta(t_2))) - g(t_1, x(\beta(t_1)))\|}{\Gamma(q)} \int_0^{\eta(t_2)} \frac{\nu(s)}{(t_2-s)^{1-q}}\,ds \\
&\qquad + \frac{e^{\delta t_2} r\zeta(t_1)}{\Gamma(q)} \biggl\{ \int_0^{\eta(t_1)-\varepsilon} \biggl(\frac{\|K(t_1,s) - K(t_2,s)\|_{\mathcal{L}(X)}}{(t_1-s)^{1-q}} + \overline{k}\biggl[\frac{1}{(t_1-s)^{1-q}} - \frac{1}{(t_2-s)^{1-q}}\biggr]\biggr) \\
&\qquad \times \nu(s)\,ds + \overline{k}\int_{\eta(t_1)-\varepsilon}^{\eta(t_1)} \biggl(\frac{\nu(s)}{(t_1-s)^{1-q}} + \frac{\nu(s)}{(t_2-s)^{1-q}}\biggr)\,ds + \overline{k}\int_{\eta(t_1)}^{\eta(t_2)} \frac{\nu(s)}{(t_2-s)^{1-q}}\,ds \biggr\}.
\end{aligned}
$$

Under the conditions (H3) and (H4), $\|(\mathcal{G}x)(t_2) - (\mathcal{G}x)(t_1)\| \to 0$ as $t_2 \to t_1$ and $\varepsilon \to 0$, that is, the set $\mathcal{G}(B_r)$ is equicontinuous.

Next, we prove that the set $U(t) = \{\mathcal{G}x(t) : x \in B_r, t \in [0,a]\}$ is a relatively compact subset of X for every $a \in (0, +\infty)$. For arbitrary $\varepsilon \in (0, \eta(t))$, define an operator $\mathcal{G}_\varepsilon$ as follows:

$$
(\mathcal{G}_\varepsilon x)(t) = \frac{g(t, x(\beta(t)))}{\Gamma(q)} \int_0^{\eta(t)-\varepsilon} \frac{K(t,s)h(s, x(\gamma(s)))}{(t-s)^{1-q}}\,ds.
$$

Noting that (H5)(c) and (H3), for $x \in B_r$, from the mean value theorem for the Bochner integral (see [17], Lemma 2.1.3), we obtain

$$
(\mathcal{G}_\varepsilon x)(t) \in \frac{(t-\varepsilon)\tilde{\zeta}_a}{\Gamma(q)} \overline{\operatorname{co}\bigl\{(t-s)^{q-1}K(t,s)h(s,x) : s \in [0, t-\varepsilon], x \in B_r\bigr\}},
$$

where $\operatorname{co}(\cdot)$ denotes the convex hull. Using (H5)(d), we infer that the set $\{\mathcal{G}_\varepsilon x(t) : x \in B_r\}$ is relatively compact in X for arbitrary $\varepsilon \in (0, \eta(t))$.

Moreover, for $x \in B_r$, $t \in [0,a]$, we get

$$
\begin{aligned}
\bigl\|(\mathcal{G}x)(t) - (\mathcal{G}_\varepsilon x)(t)\bigr\| &= \frac{\|g(t, x(\beta(t)))\|}{\Gamma(q)} \int_{\eta(t)-\varepsilon}^{\eta(t)} \frac{\|K(t,s)h(s, x(\gamma(s)))\|}{(t-s)^{1-q}}\,ds \\
&\leq \frac{e^{\delta a} r\tilde{\zeta}_a\overline{k}}{\Gamma(q)} \int_{\eta(t)-\varepsilon}^{\eta(t)} \frac{\nu(s)}{(t-s)^{1-q}}\,ds.
\end{aligned}
$$

Hence, there are relatively compact sets arbitrarily close to the set $U(t)$, $t \geq 0$. This proves that the set $U(t)$, $t \geq 0$ is relatively compact in X.

Moreover, for $x \in B_r$, we obtain

$$e^{-\delta t}\big\|(\mathcal{G}x)(t)\big\| \leq \frac{\overline{k}\zeta(t)r}{\Gamma(q)}\int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \to 0, \quad t \to \infty.$$

Now, from Lemma 3.1, we can conclude that $\mathcal{G}(B_r)$ is relatively compact in $C_\delta^0(X)$. Thus, $\mathcal{G}$ is completely continuous.

Next, we prove that there exists $r > 0$ such that $\mathcal{M}B_r \subset B_r$, where $B_r = B_r(0, C_\delta^0(X))$. Suppose on the contrary that, for each $r > 0$, there exist $x^* \in B_r$ and some $t^* \geq 0$ such that $e^{-\delta t^*}\|(\mathcal{M}x^*)(t^*)\| > r$. Then

$$\begin{aligned} r &< e^{-\delta t^*}\big\|\big(\mathcal{M}x^*\big)\big(t^*\big)\big\| \\ &\leq L_f\big(t^*\big)\big\|x^*\big\|_\delta + e^{-\delta t^*}\big\|f\big(t^*,0\big)\big\| + \frac{\zeta(t^*)\overline{k}}{\Gamma(q)}\int_0^{\eta(t^*)} \frac{\nu(s)e^{-\delta(t^*-s)}}{(t^*-s)^{1-q}}\,ds \cdot \big\|x^*\big\|_\delta \\ &\leq \left(L_f\big(t^*\big) + \frac{\zeta(t^*)\overline{k}}{\Gamma(q)}\int_0^{\eta(t^*)} \frac{\nu(s)e^{-\delta(t^*-s)}}{(t^*-s)^{1-q}}\,ds\right)r + e^{-\delta t^*}\big\|f\big(t^*,0\big)\big\|. \end{aligned} \tag{3.9}$$

Dividing both sides of (3.9) by r and taking $r \to \infty$, we obtain

$$L_f\big(t^*\big) + \frac{\zeta(t^*)\overline{k}}{\Gamma(q)}\int_0^{\eta(t^*)} \frac{\nu(s)e^{-\delta(t^*-s)}}{(t^*-s)^{1-q}}\,ds \geq 1.$$

This contradicts (3.1). This shows that there exists $r > 0$ such that $\mathcal{M}$ is a condensing map from B_r into B_r. Now from Lemma 2.5 we see that the operator $\mathcal{M}$ has a fixed point and thus equation (1.1) has at least one solution on $C_\delta^0(X)$.

Moreover, the solutions of equation (1.1) are lying in the ball B_r, and for any solutions x, y of equation (1.1) and $x, y \in B_r$, we obtain

$$\begin{aligned} e^{-\delta t}\big\|x(t) - y(t)\big\| \leq{}& L_f(t)\big(\|x\|_\delta + \|y\|_\delta\big) \\ &+ \frac{\overline{k}\|x\|_\delta}{\Gamma(q)}\big\|g\big(t, x\big(\beta(t)\big)\big) - g\big(t, y\big(\beta(t)\big)\big)\big\|\int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \\ &+ \frac{\overline{k}\zeta(t)e^{-\delta t}}{\Gamma(q)}\int_0^{\eta(t)} \frac{\|h(s, x(\gamma(s))) - h(s, y(\gamma(s)))\|}{(t-s)^{1-q}}\,ds \\ \leq{}& 2rL_f(t) + \frac{4r\overline{k}\zeta(t)}{\Gamma(q)}\int_0^{\eta(t)} \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds \to 0, \quad t \to \infty, \end{aligned}$$

then the solutions of equation (1.1) are uniformly locally attractive by Definition 2.6 (or equivalently that solutions of (1.1) are asymptotically stable). □

From the proof of Theorem 3.2, we can immediately study equation (1.1) on $X = \mathbf{R}$ under the following assumptions:

(H1′) $\alpha, \beta, \gamma, \eta : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, η is nondecreasing on $\mathbf{R}_+$, $\alpha(t) \to \infty$, $\eta(t) \to \infty$, as $t \to \infty$ and $\alpha(t), \eta(t), \gamma(t) \leq t$.

(H2′) The function $f(t,x(\alpha(t)))$ is continuous with respect to t on $[0,+\infty)$ and there exists a continuous function $L_f(t)$ such that

$$\bigl|f(t,\psi_1)-f(t,\psi_2)\bigr| \le L_f(t)|\psi_1-\psi_2|, \quad \text{for } \psi_1,\psi_2 \in C(\mathbf{R}_+,\mathbf{R}),$$

where $L_f^* = \sup_{t\ge 0} L_f(t) < 1$, $\lim_{t\to\infty} L_f(t) = 0$, and $\sup_{t\ge 0} |f(t,0)| < \infty$.

(H3′) The function $g(t,x(\beta(t)))$ is continuous with respect to t on $[0,+\infty)$ and for every $a>0$, $g(t,\cdot):\mathbf{R}\to\mathbf{R}$ is continuous for $t\in[0,a]$. The set $V(r,g)$ is an equicontinuous subset of $C(\mathbf{R}_+,\mathbf{R})$ and there exists a function $\zeta:\mathbf{R}_+\to\mathbf{R}_+$ such that $|g(t,x)|\le\zeta(t)$ and $\sup_{t\in[0,a]}\zeta(t)=\tilde{\zeta}_a<\infty$, for every $a>0$.

(H4′) The function $K:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}$ is continuous and $|K(t,s)|\le\overline{k}$.

(H5′) For every $a>0$, the function $h:[0,a]\times\mathbf{R}\to\mathbf{R}$ satisfies the following conditions:
  (a) the function $h(t,\cdot):\mathbf{R}\to\mathbf{R}$ is continuous a.e. $t\in[0,a]$;
  (b) the function $h(\cdot,x):[0,a]\to\mathbf{R}$ is strongly measurable for every $x\in\mathbf{R}$;
  (c) there exists $\nu(\cdot)\in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that $|h(t,x)|\le\nu(t)|x|$, the function $s\to\frac{\nu(s)}{(t-s)^{1-q}}$ belongs to $L^1([0,t],\mathbf{R}_+)$ and

$$\lim_{t\to\infty}\zeta(t)\int_0^{\eta(t)}\frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds=0.$$

We choose $\delta>0$ such that (3.1) holds, then we have the following result.

Theorem 3.3 *Assume that* (H1′)-(H5′) *hold, then there exists a solution on the space* $C_\delta^0(\mathbf{R})$ *for equation* (1.1). *Moreover, the solutions of equation* (1.1) *are uniformly locally attractive* (*or, equivalently, the solutions are asymptotically stable*).

3.2 Application to fractional differential equations

Motivated by the proof of Theorem 3.2, we can immediately obtain the global existence of a mild solution for the fractional differential equation as follows:

$$\begin{cases} {}^{c}D_t^q(x(t)-m(t,x(t))) = A(x(t)-m(t,x(t)))+h(t,x(t)), & t>0,\\ x(0)=x_0, \end{cases} \tag{3.10}$$

where $0<q<1$, $m(0,x(0))=0$, A is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\ge0}$ in X with $\|T(t)\|\le M$.

It is well known that a function $x\in C(\mathbf{R}_+,X)$ and

$$x(t)=Q(t)x_0+m\bigl(t,x(t)\bigr)+\int_0^t(t-s)^{q-1}R(t-s)h\bigl(s,x(s)\bigr)\,ds,\quad t\ge0, \tag{3.11}$$

is the mild solution of (3.10), where

$$Q(t)=\int_0^\infty\xi_q(\sigma)T\bigl(t^q\sigma\bigr)\,d\sigma \quad\text{and}\quad \bigl\|Q(t)\bigr\|\le M,$$
$$R(t)=q\int_0^\infty\sigma\xi_q(\sigma)T\bigl(t^q\sigma\bigr)\,d\sigma \quad\text{and}\quad \bigl\|R(t)\bigr\|\le\frac{M}{\Gamma(q)},$$

and ξ_q is a probability density function defined on $(0,\infty)$ (see [6]) such that

$$\xi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{n-1}\frac{\Gamma(nq)}{(n-1)!}\sin(n\pi q) \geq 0, \quad \sigma \in (0,\infty).$$

For more details, we refer to [6, 7].

Set $f(t,x) = Q(t)x_0 + m(t,x(t)), K(t-s) = R(t-s)$, we can study the existence of solution for equation (3.11) by Theorem 3.2. To this end, for every $a > 0$, $\delta > 0$, we make the following assumptions:

(A1) The function $m(t,x(t))$ is continuous with respect to t on $[0,+\infty)$ and there exists a continuous function $m_f(t)$ such that

$$\|m(t,\psi_1) - m(t,\psi_2)\| \leq m_f(t)\|\psi_1 - \psi_2\|, \quad \text{for } \psi_1,\psi_2 \in C(\mathbf{R}_+,X),$$

where $m_f^* = \sup_{t\geq 0} m_f(t) < 1$, $\lim_{t\to\infty} m_f(t) = 0$, and $\sup_{t\geq 0}\|m(t,0)\| < \infty$.

(A2) For every $a > 0$, the function $h : [0,a] \times X \to X$ satisfies the following conditions:
   (a) the function $h(t,\cdot) : X \to X$ is continuous a.e. $t \in [0,a]$;
   (b) the function $h(\cdot,x) : [0,a] \to X$ is strongly measurable for every $x \in X$;
   (c) there exists $\nu(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that $\|h(t,x)\| \leq \nu(t)\|x\|$, the function $s \to \frac{\nu(s)}{(t-s)^{1-q}}$ belongs to $L^1([0,t],\mathbf{R}_+)$ and $\lim_{t\to\infty}\int_0^t \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds = 0$;
   (d) for every $t > 0$ and $r > 0$, the set $\{T(t-s)h(s,e^{\delta s}z) : s \in [0,t), z \in B_r(0,C_\delta^0(X))\}$ is relatively compact in X.

From the continuity of $T(t)$ in the uniform operator topology, (A1) and (A2), it is easy to check that the assumptions in Theorem 3.2 hold. If we choose $\delta > 0$ such that

$$m_f^* + \sup_{t\geq 0}\frac{M}{\Gamma(q)}\int_0^t \frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds < 1,$$

then we obtain the following result on the space $C_\delta^0(X)$.

Theorem 3.4 *Assume that* (A1) *and* (A2) *are satisfied, then there exists a global mild solution for* (3.10) *on the space* $C_\delta^0(X)$. *Moreover, the mild solutions of* (3.10) *are uniformly locally attractive* (*or equivalently, the solutions are asymptotically stable*).

Next, we consider the following fractional differential equation in $X = \mathbf{R}$:

$$\begin{cases} {}^{c}D_t^q x(t) = h(t,x(t)), & t > 0, \\ x(0) = x_0. \end{cases} \tag{3.12}$$

If $h(t,x(t))$ is continuous with respect to t on $\mathbf{R}_+$, from Remark 2.4,

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h\big(s,x(s)\big)\,ds$$

is equivalent to equation (3.12). For more details see [5] and the references therein.

For every $a > 0$, $\delta > 0$, assume the function $h : [0,a] \times \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:

(a) the function $h(t,\cdot):\mathbf{R}\to\mathbf{R}$ is continuous a.e. $t\in[0,a]$;
(b) the function $h(\cdot,x):[0,a]\to\mathbf{R}$ is strongly measurable for every $x\in\mathbf{R}$;
(c) there exists $\nu(\cdot)\in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that $|h(t,x)|\le\nu(t)|x|$, the function $s\to\frac{\nu(s)}{(t-s)^{1-q}}$ belongs to $L^1([0,t],\mathbf{R}_+)$ and $\lim_{t\to\infty}\int_0^t\frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds=0$.

Using the results of Theorem 3.2, and choosing δ such that

$$\sup_{t\ge 0}\frac{1}{\Gamma(q)}\int_0^t\frac{\nu(s)e^{-\delta(t-s)}}{(t-s)^{1-q}}\,ds<1,$$

we can obtain without proof the following result on the space $C^0_\delta(\mathbf{R})$.

Theorem 3.5 *Assume that* (a), (b), (c) *are satisfied, then on the space* $C^0_\delta(\mathbf{R})$, (3.12) *has a global solution which is uniformly locally attractive* (*or, equivalently, the solution is asymptotically stable*).

4 Examples

As applications of our results, we study the following examples.

Example 4.1 Let $X=L^2([0,\pi])$, we consider the following fractional integral equation:

$$\begin{aligned} x(t,\xi)={}&\frac{t^2\sin(x(t,\xi))}{1+t^4}+\left(\frac{\int_0^t s^{-\frac{1}{2}}\cos(x(s,\xi))\,ds}{40(t+2)^{\frac{1}{4}}\Gamma(\frac{1}{2})}\right)\\ &\times\int_0^{\frac{t}{2}}\frac{e^{-t-3s}h(s,x(\frac{s}{3},\xi))}{(t-s)^{\frac{1}{2}}}\,ds,\quad t\ge 0, \end{aligned}\tag{4.1}$$

where $\alpha(t)=\beta(t)=t$, $\eta(t)=\frac{t}{2}$, $\gamma(t)=\frac{t}{3}$, $K(t,s)=e^{-t-3s}$, $q=\frac{1}{2}$, and

$$h\left(t,x\left(\frac{t}{3},\xi\right)\right)=(1+t)^{-\frac{1}{2}}\sin\left(x\left(\frac{t}{3},\xi\right)\right)\int_0^\pi e^{-|x(\frac{\xi}{3},\xi)|}\,d\xi.$$

Observe that this equation has the form of equation (1.1) if we put

$$\begin{aligned} &x(t)(\xi)=x(t,\xi),\\ &f\bigl(t,x\bigl(\alpha(t)\bigr)(\xi)\bigr)=\frac{t^2\sin(x(t)(\xi))}{1+t^4},\\ &g\bigl(t,x\bigl(\beta(t)\bigr)(\xi)\bigr)=\frac{1}{40}(t+2)^{-\frac{1}{4}}\int_0^t s^{-\frac{1}{2}}\cos\bigl(x(s)(\xi)\bigr)\,ds,\\ &h\bigl(t,x\bigl(\gamma(t)\bigr)(\xi)\bigr)=(1+t)^{-\frac{1}{2}}\sin\left(x\left(\frac{t}{3}\right)(\xi)\right)\int_0^\pi e^{-|x(\frac{\xi}{3})(\xi)|}\,d\xi. \end{aligned}$$

We can easily see

$$\begin{aligned} &\bigl\|f(t,\psi_1)-f(t,\psi_2)\bigr\|\le\frac{t^2}{1+t^4}\bigl\|\psi_1(t)(\xi)-\psi_2(t)(\xi)\bigr\|,\quad \psi_1,\psi_2\in C(\mathbf{R}_+,X),\\ &\bigl\|g(t,x)\bigr\|\le\frac{1}{20}(t+2)^{-\frac{1}{4}}t^{\frac{1}{2}},\\ &\bigl\|h(t,x)\bigr\|<\pi t^{-\frac{1}{2}}\|x\|. \end{aligned}$$

Obviously, $\lim_{t\to\infty}\frac{t^2}{1+t^4}=0$ and $\frac{t^2}{1+t^4}\leq\frac{1}{2}$, the function $s\to\frac{s^{-\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}$ belongs to $L^1([0,t],\mathbf{R}_+)$ and, for any $\delta>0$, from Lemma 2.8(2),

$$\frac{\pi t^{\frac{1}{2}}}{20(t+2)^{\frac{1}{4}}}\int_0^{\frac{t}{2}} s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\,ds\leq\frac{\sqrt{2}\pi\delta^{-\frac{1}{2}}}{10(t+2)^{\frac{1}{4}}}\Gamma\left(\frac{1}{2}\right)\to 0,\quad t\to\infty.$$

Now, we can see clearly that (H1)-(H5) in Theorem 3.2 hold for $x\in C(\mathbf{R}_+,X)$.

Let $\delta=1$, we have

$$\frac{1}{2}+\sup_{t\geq 0}\frac{\pi t^{\frac{1}{2}}}{20(t+2)^{\frac{1}{4}}\Gamma(\frac{1}{2})}\int_0^{\frac{t}{2}}\frac{s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}}{e^{t-s}}\,ds\leq\frac{1}{2}+\frac{\sqrt{2}\pi}{10}\sup_{t\geq 0}\frac{1}{(t+2)^{\frac{1}{4}}}\approx 0.874<1,$$

then equation (4.1) has a solution on the space $C_1^0(X)$ by Theorem 3.2 and the solutions of (4.1) are uniformly locally attractive by Theorem 3.2.

Example 4.2 Let $X=L^2([0,\pi])$, we consider the following fractional heat conduction equation:

$$\begin{cases}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}(\nu(t,\xi)-\frac{t\sin(\nu(t,\xi))}{10(1+t^2)})=\frac{\partial^2}{\partial\xi^2}(\nu(t,\xi)-\frac{t\sin(\nu(t,\xi))}{10(1+t^2)})+e^{-t}\sin(\frac{\nu(t,\xi)}{\sqrt[4]{t}}), & t>0,\\ \nu(t,0)-\frac{t\sin(\nu(t,0))}{10(1+t^2)}=\nu(t,\pi)-\frac{t\sin(\nu(t,\pi))}{10(1+t^2)}=0, & t>0,\\ \nu(0,\xi)=\nu_0.\end{cases}\tag{4.2}$$

To treat the above problem, we define $D(A)=H^2([0,\pi])\cap H_0^1([0,\pi])$, $Au=-u''$. The operator $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X and $\{T(t)\}_{t>0}$ is compact and $\|T(t)\|\leq 1$.

For $\xi\in[0,\pi]$, we set

$$x(t)(\xi)=\nu(t,\xi),$$
$$h(t,x)(\xi)=e^{-t}\sin\left(\frac{x(t)(\xi)}{\sqrt[4]{t}}\right),$$
$$m(t,x)(\xi)=\frac{t\sin(x(t)(\xi))}{10(1+t^2)}.$$

Then the above equation (4.2) can be reformulated as the abstract equation (3.10).

Clearly, for $t>0$, $\psi_1,\psi_2\in C(\mathbf{R}_+,X)$,

$$\left\|m(t,\psi_1)-m(t,\psi_2)\right\|\leq\frac{t}{10(1+t^2)}\|\psi_1-\psi_2\|,$$

where $\lim_{t\to\infty}\frac{t}{10(1+t^2)}=0$ and $\frac{t}{1+t^2}\leq\frac{1}{2}$. Moreover, $\|h(t,x)\|\leq t^{-\frac{1}{4}}e^{-t}\|x\|$ and the function $s\to\frac{s^{-\frac{1}{4}}e^{-s}}{(t-s)^{\frac{1}{2}}}$ belongs to $L^1([0,t],\mathbf{R}_+)$. Moreover, for any $\delta>1$, noting that Lemma 2.8(2) holds, we have

$$\int_0^t\frac{s^{-\frac{1}{4}}e^{-s}e^{-\delta(t-s)}}{(t-s)^{\frac{1}{2}}}\,ds=\frac{1}{e^t}\int_0^t\frac{s^{-\frac{1}{4}}e^{-(\delta-1)(t-s)}}{(t-s)^{\frac{1}{2}}}\,ds\leq\frac{5\sqrt[4]{2}\Gamma(\frac{1}{2})}{3e^t\sqrt[4]{t}(\delta-1)^{\frac{1}{2}}}\to 0,\quad t\to\infty.$$

For $\delta \in (0,1]$, by Lemma 2.8(1), we obtain

$$\begin{aligned} \int_0^t \frac{s^{-\frac{1}{4}} e^{-s} e^{-\delta(t-s)}}{(t-s)^{\frac{1}{2}}} \, ds &= \frac{1}{e^{\delta t}} \int_0^t \frac{s^{-\frac{1}{4}} e^{-(1-\delta)s}}{(t-s)^{\frac{1}{2}}} \, ds \leq \frac{t^{\frac{1}{4}}}{e^{\delta t}} B\left(\frac{3}{4}, \frac{1}{2}\right) \\ &\to 0, \quad t \to \infty. \end{aligned} \tag{4.3}$$

Hence, for any $\delta > 0$, $\lim_{t\to\infty} \int_0^t \frac{s^{-\frac{1}{4}} e^{-s} e^{-\delta(t-s)}}{(t-s)^{\frac{1}{2}}} \, ds = 0$. Moreover, (A2)(d) is ensured by the compactness of $\{T(t)\}_{t>0}$. Now, (A1) and (A2) in Theorem 3.4 hold for $x \in C(\mathbf{R}_+, X)$. Choosing $\delta = 1$, from (4.3), we obtain

$$\frac{1}{20} + \sup_{t\geq 0} \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{s^{-\frac{1}{4}} e^{-s} e^{-(t-s)}}{(t-s)^{\frac{1}{2}}} \, ds \leq \frac{1}{20} + \frac{4\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sup_{t\geq 0} \frac{t^{\frac{1}{4}}}{e^t} \approx 0.795 < 1.$$

Then the existence and uniformly local attractivity of the global mild solution of (4.2) on the space $C_1^0(X)$ can be obtained by Theorem 3.4.

Example 4.3 Let $X = \mathbf{R}$, we consider the following fractional differential equation:

$$\begin{cases} {}^{c}D_t^{\frac{1}{4}} x(t) = \frac{1}{4}(t^{-\frac{1}{4}} + 1)e^{-2t} \sin(x(t)), \quad t > 0, \\ x(0) = 1. \end{cases} \tag{4.4}$$

Clearly, if $x(t) \in C(\mathbf{R}_+, \mathbf{R})$, the integral equation

$$x(t) = 1 + \frac{1}{\Gamma(\frac{1}{4})} \int_0^t \frac{h(s, x(s))}{(t-s)^{\frac{3}{4}}} \, ds$$

is the solution of (4.4), where $h(t, x(t)) = \frac{1}{4}(t^{-\frac{1}{4}} + 1)e^{-2t} \sin(x(t))$, by simple calculations we see that $|h(t,x)| \leq \frac{1}{4}(t^{-\frac{1}{4}} + 1)e^{-2t}|x|$ and the function $s \to \frac{(s^{-\frac{1}{4}}+1)e^{-2s}}{(t-s)^{\frac{3}{4}}}$ belongs to $L^1([0,t], \mathbf{R}_+)$. Moreover, for any $\delta > 2$, noting that Lemma 2.8(1) holds, we have

$$\begin{aligned} \int_0^t \frac{(s^{-\frac{1}{4}} + 1)e^{-2s} e^{-\delta(t-s)}}{(t-s)^{\frac{3}{4}}} \, ds &= \frac{1}{e^{2t}} \left[\int_0^t \frac{s^{-\frac{1}{4}} e^{-(\delta-2)(t-s)}}{(t-s)^{\frac{3}{4}}} \, ds + \int_0^t \frac{e^{-(\delta-2)(t-s)}}{(t-s)^{\frac{3}{4}}} \, ds \right] \\ &\leq \frac{1}{e^{2t}} \left[B\left(\frac{3}{4}, \frac{1}{4}\right) + 4t^{\frac{1}{4}} \right] \\ &\to 0, \quad t \to \infty. \end{aligned} \tag{4.5}$$

For $\delta \in (0,2]$, by Lemma 2.8(1),

$$\begin{aligned} \int_0^t \frac{(s^{-\frac{1}{4}} + 1)e^{-2s} e^{-\delta(t-s)}}{(t-s)^{\frac{3}{4}}} \, ds &= \frac{1}{e^{\delta t}} \int_0^t \frac{(s^{-\frac{1}{4}} + 1)e^{-(2-\delta)s}}{(t-s)^{\frac{3}{4}}} \, ds \\ &\leq \frac{1}{e^{\delta t}} \left[B\left(\frac{3}{4}, \frac{1}{4}\right) + 4t^{\frac{1}{4}} \right] \\ &\to 0, \quad t \to \infty. \end{aligned}$$

Hence, for any $\delta > 0$, $\int_0^t \frac{(s^{-\frac{1}{4}}+1)e^{-2s}e^{-\delta(t-s)}}{(t-s)^{\frac{3}{4}}}\,ds \to 0$, $t \to \infty$. It is now obvious that conditions (a), (b), (c) in Theorem 3.5 hold for $x \in C(\mathbf{R}_+, \mathbf{R})$. By choosing $\delta = 3$, from (4.5), we have

$$\sup_{t\geq 0} \frac{1}{4\Gamma(\frac{1}{4})} \int_0^t \frac{(s^{-\frac{1}{4}}+1)e^{-2s}e^{-3(t-s)}}{(t-s)^{\frac{3}{4}}}\,ds \leq \frac{1}{4}\Gamma\left(\frac{3}{4}\right) + \frac{1}{\Gamma(\frac{1}{4})} \sup_{t\geq 0} \frac{t^{\frac{1}{4}}}{e^{2t}} \approx 0.434 < 1.$$

Then the existence and uniformly local attractivity of the solution of equation (4.4) on the space $C_3^0(X)$ can be obtained by Theorem 3.5.

5 Conclusions

We discuss the existence and attractivity of global solutions for a class of nonlinear fractional quadratic integral equations in a Banach space X in this paper. The nonlinear fractional-order quadratic integral equation on an unbounded interval is difficult to solve. By employing some necessary restrictions on nonlinear terms and defining a Banach space $C_\delta^0(X)$, we obtain the existence and attractivity results of global solutions on $C_\delta^0(X)$, these strategies are different from that of [10–16]. As an application, we obtain the corresponding results for the global mild solutions of two classes of fractional differential equations. The above mentioned techniques are new even in the case of $X = \mathbf{R}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, HW and FL, contributed to each part of this study equally and read and approved the final version of the manuscript.

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